11/03 - Kempf's theorem

Note Title 11/3/2016

Last time: Ended by saying if $X \to \mathrm{Spec}(\Gamma(\mathcal{O}_X))$ is projective and G acts on X linearizably, i.e. $X \hookrightarrow \mathbb{P}^M \times \mathbb{A}^N$ with G acting, then call $\mathcal{L} = \mathcal{O}_X(1)$

A) $\forall f \in \Gamma(X, \mathcal{L}^n)^G$, $n>0$, $X_f = \{x \in X \mid f(x) \neq 0\}$ is still affine

$$X^{ss}(\mathcal{L})/G = \bigcup_{\substack{n \geq 1 \\ f \in \Gamma(\mathcal{L}^n)^G}} X_f/G \xrightarrow{q} \mathrm{Proj} \bigoplus_{n \geq 0} \Gamma(\mathcal{L}^n)^G$$

$$\mathrm{Proj} \bigoplus_{n \geq 0} \Gamma(\mathcal{L}^n)^G \xrightarrow{\text{projective}} \mathrm{Spec}\, \Gamma(\mathcal{O}_X)^G$$

q is a GMS

B) the HM criterion holds as stated

Consequences of HM criterion:

(will come back to this)

2) $X^{ss}(\mathcal{L})$ only depends on $c_1(\mathcal{L}) \in H^2_G(X;\mathbb{Q})$

1) can define $X^{ss}(\ell)$ for $\ell \in \mathrm{Pic}(X/G) \otimes \mathbb{Q}$

3) if you perturb $\ell \rightsquigarrow \ell + \epsilon \ell'$ for ϵ small, then $X^{ss}(\ell + \epsilon\ell') \subset X^{ss}(\ell)$

4) If $Y \xrightarrow{\pi} X$ is finite, then

$$Y^{ss}(\ell) = \pi^{-1}(X^{ss}(\ell))$$

Ex: $Y = (\mathbb{P}^1)^n \longrightarrow \mathbb{P}^n \circlearrowleft SL_2$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(r_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^1}(r_n)$

$\lambda : \mathbb{G}_m \longrightarrow SL_2$ is a choice of basis up to scaling $\mathbb{P}^1 = \{[x:y]\}$

$t \cdot [x:y] = [t^{-1}x : ty]$

$Y =$ n-tuples of linear forms

$Y^{ss}(\mathcal{L}) = \{(\ell_1, .., \ell_n) \mid \forall \ell, \sum_{\ell_i = \ell} r_i \leq \sum_{\ell_i \neq \ell} r_i\}$

The symmetric linearization desends to $\mathbb{P}^n$

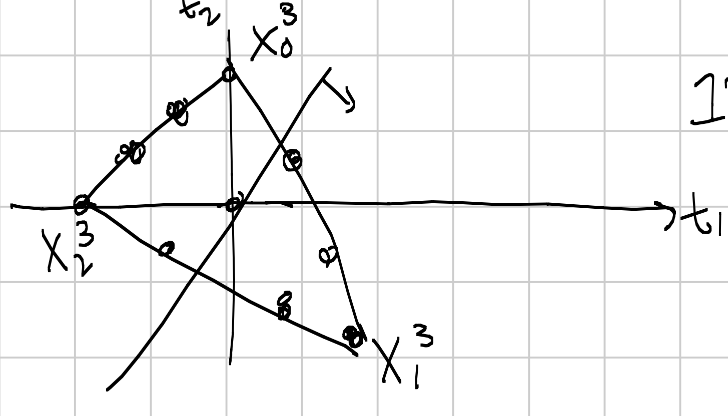
$\rightsquigarrow$ binary form $f(x,y)$ is semistable iff has no root of multiplicity $> \frac{n}{2}$

$\boxed{\text{Can consider the dependence of } Y^{ss} \text{ on } r_1, .., r_n}$

$\hookrightarrow$ VGIT, can define stability for classes in $\mathrm{Pic} \otimes \mathbb{R}$

Ex: $SL_3 \circlearrowleft \mathbb{P}(\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(2))) = \mathbb{P}(\mathrm{Sym}^3 \mathbb{C}^3)$

Char wrt. $T \subset SL_3$, standard max torus



1PS $\rightsquigarrow$ codirection

limit $\rightsquigarrow$

$wt_\lambda(\mathcal{O}(1)_y)$

$= -\langle \lambda, \chi_{min} \rangle$

Point is T- semistable iff $St(p) \subset \mathfrak{t}_{\mathbb{R}}$ contains origin